Math 245B Lecture 8 Notes

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1 The Stone-Weierstrass Theorem

1.1 Algebras of functions

Last time, we characterized compact (in some sense 'small' subsets of $C(X)$). This time, we will characterize larger subsets, in the sense that $\mathcal{A} \subseteq C(X)$ is dense. This also generalizes the classic Weierstrass approximation theorem.

Let X be a compact Hausdorff space. In this lecture, we will denote $C(X) = C(X, \mathbb{R})$.

Definition 1.1. A subset $\mathcal{A} \subseteq C(X)$ **separates points** if for all distinct $x, y \in X$, there exists a function $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Definition 1.2. An **algebra** of functions is a linear subspace $\mathcal{A} \subseteq C(X)$ such that if $f, g \in \mathcal{A}$, then $fg \in \mathcal{A}$.

Definition 1.3. A **lattice** of functions is a linear subspace $\mathcal{A} \subseteq C(X)$ such that if $f, g \in \mathcal{A}$, then $\max(f, g), \min(f, g) \in \mathcal{A}$.

Definition 1.4. $\mathcal{A}$ **vanishes** at $x \in X$ if $f(x) = 0$ for all $f \in \mathcal{A}$. $\mathcal{A}$ is **nowhere vanishing** if it does not vanish at any $x \in X$.

This means that for every $x \in X$, there is some $f \in \mathcal{A}$ such that $f(x) \neq 0$.

Theorem 1.1 (Stone-Weierstrass). *Let $\mathcal{A}$ be an algebra, closed under ρ_u, and separate points.*

1. *If $\mathcal{A}$ is nowhere vanishing, then $\mathcal{A} = C(X)$.*
2. *Otherwise, there exists some $x_0 \in X$ such that $\mathcal{A} = \{f \in C(X) : f(x_0) = 0\}$.*

$\mathbb{R}^2$ is an algebra over $\mathbb{R}$ with the multiplication $(x, y) \cdot (u, v) := (xu, yv)$.

Lemma 1.1. *As an algebra over $\mathbb{R}$, the only subalgebras of $\mathbb{R}^2$ are $\{(0, 0)\}$, $\{0\} \times \mathbb{R}$, $\mathbb{R} \times \{0\}$, $\{(t, t) : t \in \mathbb{R}\}$, and $\mathbb{R}^2$.*

Proof. Let A be a subalgebra of $\mathbb{R}^2$. We may assume that $\dim(A) = 1$. Let $(x, y) \in A$. Then $x^2, y^2 \in A$. These two ordered pairs must satisfy a linear relation, so $x = 0$, $y = 0$, or $x = y$. □

Remark 1.1. This is a special case of Stone-Weierstrass. If $X = \{1, 2\}$, then $C(X) = \mathbb{R}^2$.

Lemma 1.2. *There exists a sequence $(p_n)_n$ of real polynomial with $p_n(0) = 0$ such that $p_n(t) \to |t|$ uniformly for $t \in [-1, 1]$.*

Proof. Consider the Maclaurin expoansion of $\sqrt{1-s}$, where $0 \leq s < 1$. Apply this, using the fact that $|t| = +\sqrt{1-(1-t^2)}$.[1] □

Lemma 1.3. *Let $\mathcal{A}$ be a closed subalgebra of $C(X)$, Then $f \in \mathcal{A} \implies |f| \in \mathcal{A}$, and $\mathcal{A}$ is a lattice.*

Proof. Let $f \in \mathcal{A}$ with $m := \|f\|_u > 0$. By considering $|f/m| = (1/m)|f|$, we may assume that $m \leq 1$. Let $(p_n)_n$ be given by the previous lemma. Then $p_n \circ f$ converges uniformly to $|f|$ as $n \to \infty$. All the $p_n \circ f$ lie in $\mathcal{A}$, as $\mathcal{A}$ is an algebra. Since $\mathcal{A}$ is closed, $|f| \in \mathcal{A}$.

If $f, g \in \mathcal{A}$, $\max(f, g) = (1/2)|f + g| + (1/2)|f - g|$, and $\min(f, g) = -\max(-f, -g)$. So these are still in $\mathcal{A}$. □

1.2 Proof of the theorem

Now we can prove the theorem.

Proof. Suppose $\mathcal{A} \subseteq C(X)$ is a closed lattice that separates points. Also, assume $\mathcal{A}$ is nowhere vanishing.

Step 1: For all $x \neq y \in X$ consider $\mathcal{A}_{x,y} = \{(f(x), f(y)) : f \in \mathcal{A}\}$. Then $\mathcal{A}_{x,y}$ is a algebra of $\mathbb{R}^2$, separating points and nowhere vanishing. So $\mathcal{A}_{x,y} = \mathbb{R}^2$ for all x, y. Thus, for any $f \in C(X)$ and $x, y \in X$, there exists a function $g_{x,y} \in A$ such that $g - x, y(x) = f(x)$ and $g_{x,y}(y) = f(y)$.[2]

Step 2: First, here is the idea: Pin down a point x, and vary y. Each $g_{x,y}$ agrees with f at at least 2 points. Moreover, $g_x := \max_y g_{x,y}$ must satisfy $g_x(x) = f(x)$ and $g_x \geq f$ everywhere. Then we use compactness to only talk about finitely many points.

Fix $x \in X$. For all $y \in X$, we have $g_{x,y}$, as above. Fix $\varepsilon > 0$. Now there exists an open set $U_y \ni y$ such that $g_{x,y}|_{U_y} > f|_{U_y} - \varepsilon$. By compactness, there exists $X = U_{y_1} \cup \cdots \cup U_{y_m}$. Now let $g_x := ax(g - x, y_1, \ldots, g - x, y_m) \in \mathcal{A}$. We still have $g(x) = f(x)$, and for all $z \in X$¡ there exists an i such that $z \in U_{x,y_i}$. So $g_x(z) \geq g_{x,y_i}(z) > f(z) - \varepsilon$; i.e. $g_x(x) = f(x)$, and $f_x > f - \varepsilon$ everywhere.

[1]This is the way Folland proves this lemma. There are lots of equally good ways to prove this.
[2]It looks like we are using the axiom of choice here. You don't actually need it for this.

Step 3: For every $x \in X$, there exists a neighborhood $V_x \ni x$ such that $g_x|_{V_x} < f|_{V_x} + \varepsilon$. By compactness there exists a finite subcover $X = V_{x_1} \cup \cdots \cup V_{x_m}$. Let $g = \min(g_{x_1}, \dots, g_{x_m})$. Now $g < f + \varepsilon$ everywhere, and $g > f - \varepsilon$ from step 2.

If $\mathcal{A}$ vanishes at $x_0 \in X$, then it can't vanish anywhere else because it separates points. Rerun the previous proof, just altering Step 1. We know that $\mathcal{A} \subseteq \{f \in C(X) : f(x_0) = 0\}$. If $x \neq y \in X$, and $(x_0) = 0$, then we just need to show that there exists a $g_{x,y} \in \mathcal{A}$ such that $g - x, y = f(x)$ and $g_{x,y} = f(y)$. The proof is the same, except the subalgebra we get is $\mathcal{A}_{x_0,y} = \{0\} \times \mathbb{R}$. □

Theorem 1.2. *Let $\mathcal{B} \subseteq C(X)$ be an algebra that separates points. Then*

1. *If $\mathcal{B}$ is nowhere vanishing, then $\mathcal{B}$ is dense in $C(X)$.*
2. *Otherwise, there exists $x_0 \in X$ such that $\mathcal{B}$ is dense in $\{f \in C(X) : f(x_0) = 0\}$.*

Proof. Let $\mathcal{A} = \overline{\mathcal{B}}$. This is still an algebra, and we can use the other version of the theorem. □

1.3 The complex Stone-Weierstrass theorem

What about $C(X, \mathbb{C})$?

Definition 1.5. A ***-algebra** over $\mathbb{C}$ is an algebra such that $f \in \mathcal{A} \implies \overline{f} \in \mathcal{A}$.

Theorem 1.3 (complex Stone-Weierstrass)**.** *Let $\mathcal{A} \subseteq C(X, \mathbb{C})$ be a closed *-algebra that separates points. Then*

1. *If $\mathcal{B}$ is nowhere vanishing, then $\mathcal{B}$ is dense in $C(X)$.*
2. *Otherwise, there exists $x_0 \in X$ such that $\mathcal{B}$ is dense in $\{f \in C(X) : f(x_0) = 0\}$.*

Example 1.1. What if the algebra is not a *-algebra? Here is a counterexample in this case. Let $X = \{z \in \mathbb{C} : |z| = 1\}$, and let $\mathcal{A}$ be the set of complex polynomials in $C(X, \mathbb{C})$. We cannot approximate $z \mapsto \overline{z}$ by members of $\mathcal{A}$.